COMS 4995-004: Optimization for Machine Learning
Homework 4 **(Corrected version)**

HW4 is due Wednesday, Nov 27 by 1:00 pm. No late assignments will be accepted[1]. Please refer to `https://www.satyenkale.com/optml-f19/` for instructions on how to submit homework assignments.

As discussed in class, projected gradient descent makes use of a projection oracle, whereas the Frank-Wolfe method uses a linear optimization oracle. A linear optimization oracle is easier to implement than a projection oracle, and in this exercise we will formalize this qualitative statement. Suppose that K is a convex set in $\mathbb{R}^d$ in the unit ℓ_2 ball, i.e. for all $x \in K$, we have $\|x\|_2 \leq 1$. A *linear optimization oracle* for K is an algorithm that, given any vector $v \in \mathbb{R}^d$, computes $\arg\max_{x \in K} v \cdot x$. A *projection oracle* for K is an algorithm that, given any point $y \in \mathbb{R}^d$, computes $\Pi_K(y) := \arg\min_{x \in K} \|y - x\|_2^2$.

1. **(9 points)** Suppose we are given a projection oracle for K. We now want to implement an ϵ-*approximate* linear optimization oracle for K using the projection oracle, i.e. given a vector $v \in \mathbb{R}^d$, we want to find a point $x \in K$ such that $v \cdot x \geq \max_{x' \in K} v \cdot x' - \epsilon$. Show that we can find such a point x by making *one* call to the projection oracle applied to a carefully chosen point y (i.e. by computing $\Pi_K(y)$ for some point y). Give a precise formula for y in terms of v and ϵ.
   Hint: consider applying the projection oracle to a point $y = \alpha v$ for some scalar α.

2. **(16 points)** Suppose we are given a linear optimization oracle for K. We now want to implement an ϵ-*approximate* projection oracle for K using the linear optimization oracle, i.e. given a point $y \in \mathbb{R}^d$, we want to find a point $x \in K$ such that $\|y - x\|_2^2 \leq \min_{x' \in K} \|y - x'\|^2 + \epsilon$. Show that we can find such a point x by making $O(\frac{1}{\epsilon})$ calls to the linear optimization oracle. Describe your implementation of the projection oracle via pseudo-code.
   Hint: consider computing the projection via the Frank-Wolfe method.

[1] Unless you have an emergency; in that case please write to Satyen as soon as possible.